Topic —

Linear Differential Equations

B.Sc II (Math HONS)

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Study Material :→

An Equation of the form

$$\frac{dy}{dx} + Py = Q.$$

Where P and Q are functions of x or constants is said to be Linear Equation of the 1st order.

Method for finding the General solution —

Step I - First find $I.F = e^{\int P dx}$ and use $e^{x \log \ldots}$

STEP II - Multiplying both sides by $e^{\int P dx}$

STEP III
$$\frac{d\, y e^{\int P dx}}{dx} = Q e^{\int P dx}$$

$$\int d(y e^{\int P dx}) = \int Q e^{\int P dx} dx$$

$$\Rightarrow y e^{\int P dx} = \int Q e^{\int P dx} dx$$

Problem :→ solve $\frac{dy}{dx} + y \cot x = 2\cos x$

$$I.F = e^{\int \cot x\, dx} = e^{\log_e \sin x}$$
$$= e$$

Now $\sin x \frac{dy}{dx} + y \cos x = \sin x$
$$\Rightarrow = 2\cos x \sin x$$

$$\frac{d(y \sin x)}{dx} = \sin 2x$$

$$\Rightarrow \int d(y\sin x) = \int \sin 2x\, dx$$

$$\Rightarrow y\sin x = -\frac{\cos 2x}{2} + c$$

$$\Rightarrow y\sin x + \frac{\cos 2x}{2} = c$$

Ex:→ 2. Solve.

$$\cos^2 x \frac{dy}{dx} + y = \tan x$$

Solⁿ:→

Given Equation is

$$\cos^2 x \frac{dy}{dx} + y = \tan x$$

$$\Rightarrow \frac{dy}{dx} + y\sec^2 x = \frac{\sin x}{\cos x \cdot \cos^2 x}$$

$$\Rightarrow \frac{dy}{dx} + \sec^2 x \cdot y = \tan x \sec^2 x$$

Now $I.F = e^{\int \sec^2 x\, dx} = e^{\tan x}$

Now Multiplying by $e^{\tan x}$ on both sides -

$$e^{\tan x} \cdot \frac{dy}{dx} + y e^{\tan x} \sec^2 x = \tan x \sec^2 x\, e^{\tan x}$$

$$\Rightarrow \frac{d(ye^{\tan x})}{dx} = \int e^{\tan x} \cdot \tan x \sec^2 x\, dx$$

$$\Rightarrow ye^{\tan x} = \int e^{\tan x} \cdot \tan x \sec^2 x\, dx$$

Now $I_1 = \int e^{\tan x} \cdot \tan x \sec^2 x\, dx$

put $\tan x = z$

$\sec^2 x\, dx = dz$

$$I_1 = \int z e^z\, dz$$

Integrating by parts

$$z\int e^z dz - \int \int e^z dz \cdot \frac{dz}{dz} dz$$

$$= ze^z - \int e^z dz$$

$$= ze^z - e^z$$

$$= (z-1) e^z + c_1$$

$$= (\tan x - 1) e^{\tan x} + c$$

$\therefore$ Required solution.

$$ye^{\tan x} = (\tan x - 1) e^{\tan x} + c$$

$$\Rightarrow y = (\tan x - 1) + ce^{-\tan x}$$

Ex:→ 3 $\frac{dy}{dx} + \frac{1-2x}{x^2} y = 1$

Solution - Given Equation is

$$\frac{dy}{dx} + \frac{1-2x}{x^2} y = 1$$

$$\therefore I.F = e^{\int \frac{1-2x}{x^2} dx}$$

$$= e^{\int \left(\frac{1}{x^2} - \frac{2}{x}\right) dx}$$

$$= e^{\left(-\frac{1}{x} - 2\log x\right)}$$

$$= \frac{e^{-1/x}}{e^{2\log x}} = \frac{e^{-1/x}}{}$$

$$\therefore \frac{d}{dx}\frac{y e^{-1/x}}{x^2} = \frac{e^{-1/x}}{x^2}$$

$$d\left(\frac{y e^{-1/x}}{x^2}\right) = \int \frac{e^{-1/x}}{x^2}\, dx$$

$$\frac{y e^{-1/x}}{x^2} = \int \frac{e^{-1/x}}{x^2}\, dx$$

put $\frac{1}{x} = z$

$-\frac{1}{x^2}\, dx = dz$

$\frac{1}{x^2}\, dx = -dz$

$$= -\int e^{z}\, dz$$

$$= -e^{z}$$

$$= -e^{1/x} + C$$

$$\therefore \frac{y e^{-1/x}}{x^2} = -e^{1/x} + c$$

$$y e^{-1/x} = -x^2 e^{1/x} + c x^2$$

$$y = -\frac{x^2 e^{1/x}}{e^{-1/x}} + \frac{c x^2}{e^{-1/x}}$$

$$y = -x^2 e^{2/x} + c x^2 e^{1/x}$$

Which is the required solution.